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Infinite-Dimensional Widths in the Spaces of Functions, II*, †

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The concepts of three ∞-widths are proposed and some of their properties are studied in this paper. The main result is that we obtain the exact values of the three ∞-widths of Sobolev function classes $B_p^r(\mathbb{R})$ in $L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ and find the optimal subspaces and the optimal linear operator. An application of the ∞-widths to optimal recovery is given. New extremal properties of cardinal splines and cardinal spline interpolation are discovered. © 1992 Academic Press, Inc.

1. INTRODUCTION

In this paper we continue the initial work of [6] where the notions of infinite-dimensional widths both in the linear sense and in the sense of Kolmogorov were introduced. Here we will define another infinite-dimensional width in the sense of Gel'fand. For the convenience of readers, we will give the definitions and basic properties of the Kolmogorov and linear ∞-widths in Section 2. In addition, the definitions in the present paper are more general than those in [6].

The infinite-dimensional widths, abbreviated to ∞-widths, are natural extensions of n-widths. When we consider the best approximation of some classes of functions over the whole real line $\mathbb{R}$ (or the d dimensional Euclidean space $\mathbb{R}^d$), the n-widths can not work well in this situation because $\mathbb{R}$ (or $\mathbb{R}^d$) is not compact. To establish a mode for which one can compare a method of approximating a class of functions over $\mathbb{R}$ with the best possible one, we introduce the ∞-widths. Roughly speaking, the ∞-widths give the best lower bound which may be achieved by some method of approximation on some classes of functions over $\mathbb{R}$, where the

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best lower bound means the optimal order of approximation and the best constant before the order. Our main results are given in Section 3 where we obtain the exact values of three ∞-widths of Sobolev function classes $B_p^r(\mathbb{R})$ in $L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ and find the optimal subspaces and the optimal linear operator. In Section 4 we give an application of ∞-widths to the problem of optimal recovery of $B_p^r(\mathbb{R})$ in $L^p(\mathbb{R})$. It is surprising that the dilation of cardinal spline interpolation is optimal in the sense of both linear ∞-width and optimal recovery.

2. Definitions and Basic Properties

Given $w > 0$, let $\mathscr{T}_w$ be the family of spaces of functions over the real line $\mathbb{R}$ such that

$$\liminf_{a \to +\infty} \frac{1}{2a} \dim S|_{[-a,a]} \leqslant w, \qquad \text{for all} \quad S \in \mathscr{T}_w, \tag{2.1}$$

where $S|_{[-a,a]}$ is the subspace of S restricted to $[-a, a]$ and $\dim S|_{[-a,a]}$ is the dimension of $S|_{[-a,a]}$. It is clear that $S := \operatorname{span}\{\varphi(\cdot - k/w)\}_{k \in \mathbb{Z}} \in \mathscr{T}_w$ if φ is a function with compact support and $F \in \mathscr{T}_w$ if F is a finite-dimensional space of functions over $\mathbb{R}$. $\mathscr{T}_w$ contains sufficiently many spaces which are subject to the natural and reasonable condition (2.1). In the following we let $X(\mathbb{R})$ be a normed linear space of functions over $\mathbb{R}$ with norm $\|\cdot\|_X$. We usually take $X(\mathbb{R})$ as $L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ or $C^b(\mathbb{R}) := C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Definition 2.1. Let $\mathscr{T}_w$ and $X(\mathbb{R})$ be given as above and $A \subset X(\mathbb{R})$. The quantity

$$d_w(A; X(\mathbb{R})) := \inf_{S \in \mathscr{T}_w} \sup_{f \in A} \inf_{g \in S} \|f - g\|_X \tag{2.2}$$

is called the infinite-dimensional width of A in $X(\mathbb{R})$ in the sense of Kolmogorov, abbreviated ∞-K width. The ∞-linear width is defined by

$$\delta_w(A; X(\mathbb{R})) := \inf_{M} \sup_{f \in A} \|f - M(f)\|_X, \tag{2.3}$$

where the M under the inf is taken over all linear operators for which $M(\operatorname{span}(A)) \in \mathscr{T}_w$. If there exists a subspace $S^* \in \mathscr{T}_w$ such that

$$d_w(A; X(\mathbb{R})) = \sup_{f \in A} \inf_{g \in S^*} \|f - g\|_X, \tag{2.4}$$

then S^* is said to be optimal for $d_w(A; X(\mathbb{R}))$ (an optimal subspace for

$d_w(A; X(\mathbb{R})))$. Similarly, if there exists a linear operator $M^*: \operatorname{span}(A) \to M^*(\operatorname{span}(A)) \in \mathcal{T}_w$ such that

$$\delta_w(A; X(\mathbb{R})) = \sup_{f \in A} \|f - M^*(f)\|_X, \tag{2.5}$$

then M^* is called an optimal linear operator for $\delta_w(A; X(\mathbb{R}))$.

Remark. When $w = 1$, $X(\mathbb{R}) = L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, and A is the unit ball $B_p^r(\mathbb{R})$ in the Sobolev space, we have given the definitions of $d_1(B_p^r(\mathbb{R}); L^p(\mathbb{R}))$ and $\delta_1(B_p^r(\mathbb{R}); L^p(\mathbb{R}))$ in [6]. The reason why $d_w(A; X(\mathbb{R}))$ and $\delta_w(A; X(\mathbb{R}))$ are called ∞-widths is illustrated in [6]. The reason why $d_w(A; X(\mathbb{R}))$ is called the ∞-K width is that its definition is similar to that of the Kolmogorov n-width. In addition, we clearly have the relation

$$d_w(A; X(\mathbb{R})) \leqslant \delta_w(A; X(\mathbb{R})). \tag{2.6}$$

PROPOSITION 2.1. *Let $X(\mathbb{R})$ be a normed linear space of functions over $\mathbb{R}$ and $A \subset X(\mathbb{R})$. Then*

(1) $d_w(\bar{A}: X(\mathbb{R})) = d_w(A; X(\mathbb{R}))$, $\delta_w(\bar{A}; X(\mathbb{R})) = \delta_w(A; X(\mathbb{R}))$, *where $\bar{A}$ is the closed hull of A.*

(2) $d_w(\alpha A; X(\mathbb{R})) = |\alpha|\, d_w(A; X(\mathbb{R}))$, $\delta_w(\alpha A; X(\mathbb{R})) = |\alpha|\, \delta_w(A; X(\mathbb{R}))$, $\alpha \in \mathbb{R}$.

(3) $d_w(\operatorname{co}(A); X(\mathbb{R})) = d_w(A; X(\mathbb{R}))$, $\delta_w(\operatorname{co}(A); X(\mathbb{R})) = \delta_w(A; X(\mathbb{R}))$, *where* $\operatorname{co}(A)$ *denotes the convex hull of A.*

(4) *Let $b(A) := \{\alpha f: f \in A, |\alpha| \leqslant 1\}$ be the balanced hull. Then*

$$d_w(b(A); X(\mathbb{R})) = d_w(A; X(\mathbb{R})), \qquad \delta_w(b(A); X(\mathbb{R})) = \delta_w(A; X(\mathbb{R})).$$

(5) *If $w_1 < w_2$, then*

$$d_{w_2}(A; X(\mathbb{R})) \leqslant d_{w_1}(A; X(\mathbb{R})), \qquad \delta_{w_2}(A; X(\mathbb{R})) \leqslant \delta_{w_1}(A; X(\mathbb{R})).$$

(6) *If $A \subset B \subset X(\mathbb{R})$, then*

$$d_w(A; X(\mathbb{R})) \leqslant d_w(B; X(\mathbb{R})), \qquad \delta_w(A; X(\mathbb{R})) \leqslant \delta_w(B; X(\mathbb{R})).$$

The proof of Proposition 2.1 is easy, and we therefore omit it. According to the properties (1), (3), and (4), without loss of generality, we can assume that A is a closed, convex, and centrally symmetric subset of $X(\mathbb{R})$.

We now define another infinite-dimensional width which we refer to as the ∞-G width. To this end, we need to make some preparations. Let $Y(\mathbb{R})$ be a topological vector space of functions over $\mathbb{R}$. By $Y'(\mathbb{R})$ we denote the dual space which is the space of continuous linear functionals on $Y(\mathbb{R})$. In the following we define the support of an element of $Y'(\mathbb{R})$ as of the distribution [1, pp. 54–55]. For this purpose we first note that the support of a usual function f over $\mathbb{R}$ is defined by

$$\operatorname{supp} f := \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}. \tag{2.7}$$

DEFINITION 2.2. Let $\tau \in Y'(\mathbb{R})$.

(1) Suppose V is an open subset of $\mathbb{R}$. If

$$\tau(f) = 0, \qquad \text{for all} \quad f \in Y(\mathbb{R}) \quad \text{satisfying} \quad \operatorname{supp} f \subset V,$$

then τ is said to be zero on V.

(2) The support of τ is the complementary set of the largest open subset on which τ is zero. In other words, the support of τ is the smallest closed set outside of which τ is zero.

For $T := \{\tau_j\}_{j \in \mathbb{Z}}$, where $\tau_j \in Y'(\mathbb{R})$, $j \in \mathbb{Z}$, we denote $T(f) := \{\tau_j(f)\}_{j \in \mathbb{Z}}$, $f \in Y(\mathbb{R})$; $\operatorname{Ker} T := \{f \in Y(\mathbb{R}) : T(f) = 0\}$, where $T(f) = 0$ means that $\tau_j(f) = 0$, for all $j \in \mathbb{Z}$. In addition, we use the notation

$$T|_{[-a, a]} := \{\tau_j \in T : \operatorname{supp} \tau_j \cap [-a, a] \neq \varnothing\}. \tag{2.8}$$

DEFINITION 2.3. Let $w > 0$, $X(\mathbb{R})$ be a normed linear space of functions over $\mathbb{R}$, and $A \subset X(\mathbb{R})$. Set $Y_A(\mathbb{R}) := \operatorname{span}(A)$.

(1) $\Theta_w(A) := \{T = \{\tau_j\}_{j \in \mathbb{Z}} : \tau_j \in Y'_A(\mathbb{R}),\ j \in \mathbb{Z}$, and $\liminf_{a \to +\infty} (1/2a) \operatorname{card}(T|_{[-a,a]}) \leqslant w\}$, where $\operatorname{card}(B)$ stands for the cardinality of the set B.

(2) Assume $0 \in A$. The quantity

$$d^w(A; X(\mathbb{R})) := \inf_{T \in \Theta_w(A)} \sup_{f \in A \cap \operatorname{Ker} T} \|f\|_X \tag{2.9}$$

is called the infinite-dimensional width of A in $X(\mathbb{R})$ in the sense of Gel'fand, abbreviated ∞-G width. If there exists a $T^* \in \Theta_w(A)$ such that

$$d^w(A; X(\mathbb{R})) := \sup_{f \in A \cap \operatorname{Ker} T^*} \|f\|_X, \tag{2.10}$$

then $\operatorname{Ker} T^*$ is said to be an optimal subspace for $d^w(A; X(\mathbb{R}))$.

In the following we list some basic properties of $d^w(A; X(\mathbb{R}))$.

PROPOSITION 2.2. *Let $X(\mathbb{R})$ be the normed linear space of functions over $\mathbb{R}$ and $0 \in A \subset X(\mathbb{R})$.*

(1) $d^w(\alpha A; X(\mathbb{R})) = |\alpha|\, d^w(A; X(\mathbb{R})), \qquad \alpha \in \mathbb{R}.$

(2) *Let* $b(A)$ *be the balanced hull defined as in Proposition* 2.1. *Then*

$$d^w(b(A); X(\mathbb{R})) = d^w(A; X(\mathbb{R})).$$

(3) *If* $w_1 < w_2$, *then* $d^{w_2}(A; X(\mathbb{R})) \leqslant d^{w_1}(A; X(\mathbb{R}))$.

(4) *If* $A \subset B \subset X(\mathbb{R})$, *then* $d^w(A; X(\mathbb{R})) \leqslant d^w(B; X(\mathbb{R}))$.

Proof. We only prove property (4). The proof of the other properties is easy. Since $A \subset B$, $Y_A(\mathbb{R}) = \text{span}(A) \subset \text{span}(B) = Y_B(\mathbb{R})$. Thus $Y'_A(\mathbb{R}) \supset Y'_B(\mathbb{R})$. From Definition 2.3 we get $\Theta_w(A) \supset \Theta_w(B)$. Given $T \in \Theta_w(B)$, then $T \in \Theta_w(A)$ and we have clearly $B \cap \text{Ker}\, T \supset A \cap \text{Ker}\, T$. Therefore,

$$\sup_{f \in B \cap \text{Ker}\, T} \|f\|_X \geqslant \sup_{f \in A \cap \text{Ker}\, T} \|f\|_X \geqslant d^w(A; X(\mathbb{R})).$$

Since $T \in \Theta_w(B)$ is arbitrary,

$$d^w(B; X(\mathbb{R})) = \inf_{T \in \Theta_w(B)} \sup_{f \in B \cap \text{Ker}\, T} \|f\|_X \geqslant d^w(A; X(\mathbb{R})).$$

This proves (4).

Remark. (1) Similar to the case of the Gel'fand n-width, we have only

$$d^w(A; X(\mathbb{R})) \leqslant d^w(\bar{A}; X(\mathbb{R})), \qquad d^w(A; X(\mathbb{R})) \leqslant d^w(\text{co}(A); X(\mathbb{R})).$$

(2) Unlike the case of the Gel'fand n-width, we do not know whether $d^w(A; X(\mathbb{R})) \leqslant \delta_w(A; X(\mathbb{R}))$ is true in general.

3. Infinite-Dimensional Widths of $B_p^r(\mathbb{R})$ in $L^p(\mathbb{R})$

We begin this section with some notation to be used below. Let I be a finite interval or the whole real line $\mathbb{R}$. Given a $p \in [1, \infty]$ we set

$$W_p^r(I) := \{f \in L^p(I) : f^{(r-1)} \text{ loc. abs. cont. on } I \text{ and } f^{(r)} \in L^p(I)\}. \quad (3.1)$$

$W_p^r(I)$ is the usual class of Sobolev functions over I. Let

$$B_p^r(I) := \{f \in W_p^r(I) : \|f(r)\|_{L^p(I)} \leqslant 1\}, \quad (3.2)$$

where $\|h\|_{L^p(I)} := (\int_I |h(x)|^p \, dx)^{1/p}$, if $1 \leqslant p < \infty$; $:= \text{ess sup}_{x \in I} |h(x)|$, if $p = \infty$. When $I = [a, b]$ is a finite interval we denote that

$$\tilde{B}_p^r(I) := \{f \in B_p^r(I) : f^{(j)}(a) = f^{(j)}(b), j = 0, ..., r-1\}, \quad (3.3)$$

$$B_p^r(I)_0 := \{f \in \tilde{B}_p^r(I) : f^{(j)}(a) = 0, j = 0, ..., r-1\}. \quad (3.4)$$

Obviously $\tilde{B}_p^r(I)$ may be viewed as a $(b-a)$-periodic function class and

$B_p^r(I)_0$ is a subset of $\tilde{B}_p^r(I)$. Since for each $f \in B_p^r(I)_0$ we can assign zero to $f(x)$ for $x \in \mathbb{R} \backslash I$ and then $f \in B_p^r(\mathbb{R})$, $B_p^r(I)_0$ may also be viewed as a subset of $B_p^r(\mathbb{R})$ in this sense.

Let $\mathcal{S}_{r-1}$ be the space of cardinal polynomial splines of degree $r-1$ with all integers as simple knots, i.e.,

$$\mathcal{S}_{r-1} := \{s : s \in C^{r-2}(\mathbb{R}), s|_{(k,k+1)} \in \mathcal{P}_{r-1}, \text{all } k \in \mathbb{Z}\}, \tag{3.5}$$

where $\mathcal{P}_{r-1}$ is the class of polynomials of degree not exceeding $r-1$. For any bounded data $f := (f_j)_{j \in \mathbb{Z}} \in l^\infty$, it is known (cf. [3, 9]) that there is a unique bounded function $s_{r-1}(f; x) \in \mathcal{S}_{r-1}$ such that

$$s_{r-1}(f; j+\alpha_r) = f_j, \qquad \text{for all} \quad j \in \mathbb{Z},$$

where $\alpha_r := (1 + (-1)^{r-1})/4$. $s_{r-1}(f; x)$ can be expressed in the form

$$s_{r-1}(f; x) = \sum_{j \in \mathbb{Z}} f_j L(x-j), \tag{3.6}$$

where $L(x) \in \mathcal{S}_{r-1}$ satisfying $L(j+\alpha_r) = \delta_{j,0}$, $j \in \mathbb{Z}$. When $(f_j)_{j \in \mathbb{Z}}$ are the values of some function f at the points $\{j+\alpha_r\}_{j \in \mathbb{Z}}$, we also write

$$s_{r-1}(f; x) := \sum_{j \in \mathbb{Z}} f(j+\alpha_r) L(x-j). \tag{3.7}$$

The meaning of f in $s_{r-1}(f; x)$ depends on the context.

Now we are in a position to state our main results.

THEOREM 3.1. *Let r be a positive integer, $p \in [1, \infty]$, $w > 0$, and $\eta(p, r)$ be defined by*

$$\eta(p, r) := \sup\{\|f\|_{L^p[-1,1]} : f \in \tilde{B}_p^r([-1, 1]) \text{ and } f(-\cdot) = -f(\cdot) = f(\cdot+1)\}. \tag{3.8}$$

Then

$$d_w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) = \delta_w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) = d^w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) = \eta(p, r) w^{-r}.$$

Furthermore,

(1) *The following space of polynomial splines with simple knots $\{k/w\}_{k \in \mathbb{Z}}$*

$$\mathcal{S}_{r-1,w} := \left\{s(\cdot) : s\left(\frac{\cdot}{w}\right) \in \mathcal{S}_{r-1}\right\}, \tag{3.9}$$

is optimal for $d_w(B_p^r(\mathbb{R}); L^p(\mathbb{R}))$.

(2) *The interpolation operator* $s_{r-1,w}$ *defined by*

$$s_{r-1,w}(f; x) := \sum_{k \in \mathbb{Z}} f\left(\frac{k+\alpha_r}{w}\right) L(wx-k) \tag{3.10}$$

is an optimal linear operator for $\delta_w(B_p^r(\mathbb{R}); L^p(\mathbb{R}))$.

(3) $$\operatorname{Ker} T^* := \left\{ f \in W_p^r(\mathbb{R}) : f\left(\frac{k}{w}\right) = 0, \text{ all } k \in \mathbb{Z} \right\} \tag{3.11}$$

is an optimal subspace for $d^w(B_p^r(\mathbb{R}); L^p(\mathbb{R}))$.

Remark. It is easy to verify that $\eta(2, r) = \pi^{-r}$ and $\eta(1, r) = \eta(\infty, r) = \|E_r(\cdot)\|_{L^\infty(\mathbb{R})}$, where $E_r(x)$ is the Euler polynomial spline of degree r (cf. [3]), i.e., $E_r(\cdot + 1) = -E_r(\cdot)$, $E_r \in C^{r-1}(\mathbb{R})$, and $E_r^{(r)}(x) = 1$, for all $x \in (0, 1)$. In [6] we proved that $d_1(B_2^r(\mathbb{R}); L^2(\mathbb{R})) = \delta_1(B_2^r(\mathbb{R}); L^2(\mathbb{R})) = \pi^{-r}$. Besides $\mathscr{S}_{r-1}$ and s_{r-1}, since (e.g., cf. [15])

$$\|f^{(r)} - s_{2r-1}^{(r)}(f)\|_{L^2(\mathbb{R})}^2 + \|s_{2r-1}^{(r)}(f)\|_{L^2(\mathbb{R})}^2 = \|f^{(r)}\|_{L^2(\mathbb{R})}^2,$$

it follows that $\mathscr{S}_{2r-1}$ is also an optimal subspace for $d_1(B_2^r(\mathbb{R}); L^2(\mathbb{R}))$ and s_{2r-1} is also an optimal linear operator for $\delta_1(B_2^r(\mathbb{R}); L^2(\mathbb{R}))$. In addition, Sun and Li have proved in another paper [16] that when $p = 1, 2$, and ∞,

$$E(B_p^r(\mathbb{R}); \mathscr{S}_m)_p := \sup_{f \in B_p^r(\mathbb{R})} \inf_{g \in \mathscr{S}_m} \|f - g\|_{L^p(\mathbb{R})} = \eta(p, r)$$

for all integers $m \geq r-1$. These facts show that $d_w(B_p^r(\mathbb{R}); L^p(\mathbb{R}))$ may have many optimal subspaces.

The proof of Theorem 3.1 is divided into two parts: estimation from above and from below. We start with a series of lemmas and propositions which may be of some independent interest.

PROPOSITION 3.1. *Let r be a positive integer and $p \in (1, \infty)$. For each $f \in W_p^r(\mathbb{R})$, we have $s_{r-1}(f) \in W_p^r(\mathbb{R})$, and*

$$\|f - s_{r-1}(f)\|_{L^p(\mathbb{R})} \leq \eta(p, r) \|f^{(r)}\|_{L^p(\mathbb{R})}. \tag{3.12}$$

For the case $p = 2$ this proposition is proved in the recent paper [15] The proof given here is similar to that in [15] but with new lemmas. In the following lemmas, r is always a positive integer and $p \in (1, \infty)$. For convenience, we write $\sum$ or $\sum_j$ instead of $\sum_{j \in \mathbb{Z}}$ and $\int$ instead of $\int_{\mathbb{R}}$.

LEMMA 3.1. *Let $f \in W_p^r(\mathbb{R})$. Then the series $\sum |f(j+x)|^p$ converges for every $x \in \mathbb{R}$.*

Proof. Since $f \in L^p(\mathbb{R})$ and $f^{(r)} \in L^p(\mathbb{R})$ for $f \in W_p^r(\mathbb{R})$, by Stein's inequalities [12] we know that $f' \in L^p(\mathbb{R})$. Since

$$\int_0^1 \sum |f(j+x)|^p \, dx = \sum \int_0^1 |f(j+x)|^p \, dx$$
$$= \sum \int_j^{j+1} |f(x)|^p \, dx = \int |f(x)|^p \, dx < \infty,$$

$\sum |f(j+x)|^p$ converges almost everywhere. Let $x_0 \in [0, 1]$ be such that $\sum |f(j+x_0)|^p \leqslant \int |f(x)|^p \, dx < +\infty$. Then for any $x \in [0, 1]$ we have

$$\left| |f(j+x)|^p - |f(j+x_0)|^p \right| = \left| \int_{j+x_0}^{j+x} p \, |f(y)|^{p-1} f'(y) \operatorname{sgn}[f(y)] \, dy \right|$$
$$\leqslant p \int_i^{j+1} |f(y)|^{p-1} \, |f'(y)| \, dy.$$

Thus

$$\sum |f(j+x)|^p \leqslant \sum |f(j+x_0)|^p + \sum \left| |f(j+x)|^p - |f(j+x_0)|^p \right|$$
$$\leqslant \int |f(y)|^p \, dy + p \int |f(y)|^{p-1} \, |f'(y)| \, dy$$
$$\leqslant \int |f(y)|^p \, dy + p \left(\int |f(y)|^p \, dy \right)^{1/p'} \left(\int |f'(y)|^p \, dy \right)^{1/p}$$
$$=: M < \infty,$$

where $1/p' + 1/p = 1$. The inequality $\sum |f(j+x)|^p \leqslant M$ is also true for all $x \in \mathbb{R}$ since $\sum |f(j+x)|^p$ is an 1-periodic function. This proves Lemma 3.1.

LEMMA 3.2. *Suppose $f^n := (f_j^n)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ satisfies*

$$f_j^n = 0, \quad \text{for all } |j| < 2n \quad \text{and} \quad |f_j^n| \leqslant M, \quad \text{for all } |j| \geqslant 2n,$$

where $n = 1, 2, \ldots$, and M is a constant. Then

$$\lim_{n \to \infty} \|s_{r-1}(f^n)\|_{L^p[-n,n]} = 0. \tag{3.13}$$

Proof. For the fundamental function $L(x) \in \mathcal{S}_{r-1}$ appearing in (3.6), we first have to estimate $\int_{-n}^{n} |L(x-j)|^p \, dx$ for $|j| \geq 2n$. From [3] or [9] it is known that

$$|L(x)| \leqslant Ae^{-B|x|}, \qquad \text{for all} \quad x \in \mathbb{R}, \tag{3.14}$$

where A and B are positive constants depending only on r. Thus,

$$\begin{aligned}
\int_{-n}^{n} |L(x-j)|^p \, dx &\leqslant A^p \int_{-n}^{n} e^{-Bp|x-j|} \, dx \\
&= A^p e^{-Bp|j|} \int_{-n}^{n} e^{\operatorname{sgn}(j) Bpx} \, dx \\
&= \frac{A^p}{Bp} e^{-Bp|j|} (e^{Bpn} - e^{-Bpn}) \leqslant \frac{A^p e^{Bpn}}{Bp} e^{-Bp|j|}
\end{aligned} \tag{3.15}$$

for $|j| \geqslant 2n$. Hence we have

$$\begin{aligned}
&\|s_{r-1}(f^n)\|_{L^p[-n,n]} \\
&\quad \leqslant \sum_{|j| \geqslant 2n} |f_j^n| \cdot \|L(\cdot - j)\|_{L^p[-n,n]} \\
&\quad \leqslant M \sum_{|j| \geqslant 2n} \frac{Ae^{Bn}}{(Bp)^{1/p}} e^{-B|j|} = \frac{2MAe^{-Bn}}{(Bp)^{1/p}(1-e^{-B})} \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.16}$$

This proves (3.13).

For $f := (f_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ we say $f \in l^p$ provided that $\|f\|_{l^p} := (\sum |f_j|^p)^{1/p} < \infty$.

LEMMA 3.3. *Let $f \in l^p$. Then*

$$\|s_{r-1}(f)\|_{L^p(\mathbb{R})} \leqslant C \, \|f\|_{l^p} \tag{3.17}$$

with the constant $C := (\int_0^1 (\sum_k |L(x+k)|)^p \, dx)^{1/p} < \infty$.

Proof. Let $h \in L^{p'}(\mathbb{R})$ satisfying $\|h\|_{L^{p'}(\mathbb{R})} \leqslant 1$, where $1/p' + 1/p = 1$. Then

$$\begin{aligned}
&\int h(x) \, s_{r-1}(f; x) \, dx \\
&\quad \leqslant \int |h(x)| \sum |f_j| \, |L(x-j)| \, dx \\
&\quad = \sum |f_j| \int |h(x)| \, |L(x-j)| \, dx \\
&\quad = \sum |f_j| \int |h(x+j)| \, |L(x)| \, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int \left(\sum |f_j| \, |h(x+j)| \right) |L(x)| \, dx \\
&\leqslant \int \left(\sum |f_j|^p \right)^{1/p} \left(\sum |h(x+j)|^{p'} \right)^{1/p'} |L(x)| \, dx \\
&= \|f\|_{l^p} \sum_k \int_k^{k+1} |L(x)| \left(\sum_j |h(x+j)|^{p'} \right)^{1/p'} dx \\
&= \|f\|_{l^p} \sum_k \int_0^1 |L(x+k)| \left(\sum_j |h(x+k+j)|^{p'} \right)^{1/p'} dx \\
&= \|f\|_{l^p} \int_0^1 \left(\sum_k |L(x+k)| \right) \left(\sum_j |h(x+j)|^{p'} \right)^{1/p'} dx \\
&\leqslant \|f\|_{l^p} \left(\int_0^1 \left(\sum_k |L(x+k)| \right)^p dx \right)^{1/p} \left(\int_0^1 \sum_j |h(x+j)|^{p'} dx \right)^{1/p'} \\
&= \|f\|_{l^p} \left(\int_0^1 \left(\sum_k |L(x+k)| \right)^p dx \right)^{1/p} \left(\int |h(x)|^{p'} dx \right)^{1/p'} \\
&= C \, \|f\|_{l^p} \, \|h\|_{L^{p'}(\mathbb{R})} \leqslant C \, \|f\|_{l^p},
\end{aligned}
$$

where the constant C is indicated in this lemma and Hölder's inequalities are used twice. From (3.14) we know that C is a finite constant which depends only on r and p. Hence we obtain

$$
\|s_{r-1}(f)\|_{L^p(\mathbb{R})} = \sup \left\{ \int h(x) \, s_{r-1}(f; x) \, dx : \|h\|_{L^{p'}(\mathbb{R})} \leqslant 1 \right\} \leqslant C \, \|f\|_{l^p}.
$$

Remark. Professor C. A. Micchelli has told the author that Lemma 3.3 can be proved by the operator interpolation theorem. Since one can easily verify that inequality (3.17) is true for $p=1$ and $p=\infty$, (3.17) is also true for $p \in (1, \infty)$ with some constant C. However, the above direct elementary proof gives the constant C explicitly and may be of some independent interest.

LEMMA 3.4. *Let n be a positive integer. Then*

$$
\sup\{ \|f - s_{r-1}(f)\|_{L^p[-2n, 2n]} : f \in \tilde{B}_p^r([-2n, 2n]) \} = \eta(p, r),
$$

where $\eta(p, r)$ is given in (3.8).

Proof. Associate two functions f and g via the equation $g(x) = (2n/\pi)^{1/p - r} f(2nx/\pi)$. Then $g^{(r)}(x) = (2n/\pi)^{1/p} f^{(r)}(2nx/\pi)$, $\|g\|_{L^p[-\pi, \pi]} = (2n/\pi)^{-r} \|f\|_{L^p[-2n, 2n]}$, and $\|g^{(r)}\|_{L^p[-\pi, \pi]} = \|f^{(r)}\|_{L^p[-2n, 2n]}$. Thus $f \in$

$\tilde{B}_p^r([-2n, 2n])$ if and only if $g \in \tilde{B}_p^r([-\pi, \pi])$. Let $s_{r-1}^*(g; x) := (2n/\pi)^{1/p-r} s_{r-1}(f; 2nx/\pi)$ for $f \in \tilde{B}_p^r([-2n, 2n])$. Then $s_{r-1}^*(g; x)$ is a 2π-periodic polynomial spline function of degree not exceeding $r-1$, which interpolates g at the points $\{(j+\alpha_r)\pi/2n\}_{j=-2n}^{2n-1}$. By [5] we know that

$$\begin{aligned}
\sup\{\|g - s_{r-1}^*(g)\|_{L^p[-\pi,\pi]}&: g \in \tilde{B}_p^r([-\pi, \pi])\} \\
&= (2n)^{-r} \sup\{\|h\|_{L^p[-\pi,\pi]}: h \in \tilde{B}_p^r([-\pi, \pi]), h(\cdot+\pi) = -h(\cdot) = h(-\cdot)\} \\
&= \left(\frac{2n}{\pi}\right)^{-r} \sup\{\|h\|_{L^p[-1,1]}: h \in \tilde{B}_p^r([-1, 1]), h(\cdot+1) = -h(\cdot) = h(-\cdot)\} \\
&= \left(\frac{2n}{\pi}\right)^{-r} \eta(p, r).
\end{aligned}$$

Hence

$$\begin{aligned}
\sup\{\|f - s_{r-1}(f)\|_{L^p[-2n,2n]}&: f \in \tilde{B}_p^r([-2n, 2n])\} \\
&= \left(\frac{2n}{\pi}\right)^{r} \sup\{\|g - s_{r-1}^*(g)\|_{L^p[-\pi,\pi]}: g \in \tilde{B}_p^r([-\pi, \pi])\} = \eta(p, r).
\end{aligned}$$

Proof of Proposition 3.1. For $f \in W_p^r(\mathbb{R})$, Lemma 3.1 shows that $\sum_j |f(j+\alpha_r)|^p < \infty$. Therefore by Lemma 3.3, $s_{r-1}(f) \in L^p(\mathbb{R})$.

Given $\varepsilon > 0$ and noticing that $f \in W_p^r(\mathbb{R}) \subset L^p(\mathbb{R})$, there exists a number $N(\varepsilon) > 0$ such that for every $n > N(\varepsilon)$,

$$\|f - s_{r-1}(f)\|_{L^p(\mathbb{R})}^p \leqslant \varepsilon + \int_{-n}^{n} |f(x) - s_{r-1}(f; x)|^p \, dx. \tag{3.18}$$

In the following we employ Cavaretta's technique [4]. We take a function $g \in C^{r-1}(\mathbb{R})$ with the properties that $g(x) = 1$, for $|x| \leqslant 1$, $\operatorname{supp} g = [-2, 2]$, $g(x)$ is strictly monotone on $(1, 2) \cup (-2, -1)$, and $\|g^{(k)}\|_{L^\infty(\mathbb{R})} < \infty$, $k = 0, 1, ..., r$. There exist such functions [4]. Now we set

$$F_n(x) := f(x)\, g\left(\frac{x}{n}\right), \qquad x \in \mathbb{R}. \tag{3.19}$$

Then $F_n \in C^{r-1}(\mathbb{R})$, $\operatorname{supp} F_n = [-2n, 2n]$, and

$$F_n^{(r)}(x) = f^{(r)}(x)\, g\left(\frac{x}{n}\right) + \sum_{j=1}^{r} \frac{1}{n^j} \binom{r}{j} f^{(r-j)}(x)\, g^{(j)}\left(\frac{x}{n}\right).$$

Observing that $|g(x/n)| \leqslant 1$ and $|g^{(j)}(x/n)| \leqslant C_1$, all $x \in \mathbb{R}$, $j = 1, ..., r$, and from Stein's inequalities [12] that $\|f^{(r-j)}\|_{L^p(\mathbb{R})} \leqslant C_2$, $j=1, ..., r$, where C_1 and C_2 are constants independent of n, we have

$$\|F_n^{(r)}\|_{L^p[-2n,2n]} = \|F_n^{(r)}\|_{L^p(\mathbb{R})} \leqslant \|f^{(r)}\|_{L^p(\mathbb{R})} + \frac{2^r}{n} C_1 C_2. \tag{3.20}$$

Consider the periodic function $\tilde{F}_n(x)$ defined as

$$\tilde{F}_n(x) = F_n(x),\ x \in [-2n, 2n); \qquad \text{and} \qquad \tilde{F}_n(x+4n) = \tilde{F}_n(x), \quad x \in \mathbb{R}.$$

Then from $F_n^{(k)}(-2n) = F_n^{(k)}(2n) = 0$, $k=0, ..., r-1$, we know that $\tilde{F}_n/\|\tilde{F}_n^{(r)}\|_{L^p[-2n,2n]} \in \tilde{B}_p^r([-2n, 2n])$ (if $\|\tilde{F}_n^{(r)}\|_{L^p[-2n,2n]} = 0$, then $\tilde{F}_n = 0 \in \tilde{B}_p^r([-2n, 2n])$). Thus, by Lemma 3.4 and (3.20) we obtain

$$\begin{aligned}
&\|\tilde{F}_n - s_{r-1}(\tilde{F}_n)\|_{L^p[-n,n]} \\
&\quad \leqslant \|\tilde{F}_n - s_{r-1}(\tilde{F}_n)\|_{L^p[-2n,2n]} \\
&\quad \leqslant \eta(p, r)\, \|F_n^{(r)}\|_{L^p[-2n,2n]} \leqslant \eta(p, r) \left(\|f^{(r)}\|_{L^p(\mathbb{R})} + \frac{2^r}{n} C_1 C_2 \right).
\end{aligned} \tag{3.21}$$

Letting $n > N(\varepsilon)$ and noting that $\tilde{F}_n(x) = F_n(x) = f(x)$ for all $|x| \leqslant n$, the inequalities (3.18) and (3.21) yield

$$\begin{aligned}
&\|f - s_{r-1}(f)\|_{L^p(\mathbb{R})}^p \\
&\quad \leqslant \varepsilon + \|\tilde{F}_n - s_{r-1}(f)\|_{L^p[-n,n]}^p \\
&\quad \leqslant \varepsilon + (\|\tilde{F}_n - s_{r-1}(\tilde{F}_n)\|_{L^p[-n,n]} + \|s_{r-1}(\tilde{F}_n) - s_{r-1}(f)\|_{L^p[-n,n]})^p \\
&\quad \leqslant \varepsilon + \left(\eta(p, r) \left(\|f^{(r)}\|_{L^p(\mathbb{R})} + \frac{2^r C_1 C_2}{n} \right) \right. \\
&\qquad \left. + \|s_{r-1}(\tilde{F}_n) - s_{r-1}(F_n)\|_{L^p[-n,n]} + \|s_{r-1}(F_n) - s_{r-1}(f)\|_{L^p[-n,n]} \right)^p.
\end{aligned} \tag{3.22}$$

On the other hand, by Lemma 3.3, we have

$$\begin{aligned}
&\|s_{r-1}(F_n) - s_{r-1}(f)\|_{L^p[-n,n]} \\
&\quad = \|s_{r-1}(F_n - f)\|_{L^p[-n,n]} \\
&\quad \leqslant \|s_{r-1}(F_n - f)\|_{L^p(\mathbb{R})} \leqslant C \left(\sum_{j \in \mathbb{Z}} |F_n(j+\alpha_r) - f(j+\alpha_r)|^p \right)^{1/p} \\
&\quad \leqslant 2C \left(\sum_{|j| \geqslant n} |f(j+\alpha_r)|^p \right)^{1/p},
\end{aligned} \tag{3.23}$$

where the last inequality follows from the fact that $|F_n(x)| \leqslant |f(x)|$, for all $x \in \mathbb{R}$ and $F_n(x) = f(x)$, for all $|x| \leqslant n$. Since $\tilde{F}_n(j+\alpha_r) - F_n(j+\alpha_r) = 0$, $|j| < 2n$; $|\tilde{F}_n(j+\alpha_r) - F_n(j+\alpha_r)| \leqslant |\tilde{F}_n(j+\alpha_r)| \leqslant (\sum_k |f(k+\alpha_r)|^p)^{1/p} < \infty$, $|j| \geqslant 2n$, $n = 1, 2, ...,$ Lemma 3.2 gives

$$\lim_{n \to \infty} \|s_{r-1}(\tilde{F}_n) - s_{r-1}(F_n)\|_{L^p[-n,n]}$$
$$= \lim_{n \to \infty} \|s_{r-1}(\tilde{F}_n - F_n)\|_{L^p[-n,n]} = 0. \tag{3.24}$$

According to (3.23) and (3.24), letting $n \to \infty$ in (3.22), we obtain

$$\|f - s_{r-1}(f)\|^p_{L^p(\mathbb{R})} \leqslant \varepsilon + (\eta(p, r)\, \|f^{(r)}\|_{L^p(\mathbb{R})})^p.$$

Since $\varepsilon > 0$ is arbitrary, we let $\varepsilon \to 0^+$ in the above inequality and get (3.12). This completes the proof of Proposition 3.1. ∎

Remark. We should note that the inequality (3.12) is also true for the case $p = 1$ and $p = \infty$. The readers may refer to de Boor and Schoenberg [3] or Micchelli [9] for the case $p = +\infty$ and Li [7] for the case $p = 1$. In [9, 7] the general case of cardinal $\mathscr{L}$-splines is considered.

PROPOSITION 3.2. *Suppose r is a positive integer, $w > 0$, and $p \in [1, \infty]$. For $f \in W^r_p(\mathbb{R})$, let $s_{r-1,w}(f)$ and $\eta(p, r)$ be defined in* (3.10) *and* (3.8), *respectively. Then $s_{r-1,w}(f) \in L^p(\mathbb{R})$, and*

$$\|f - s_{r-1,w}(f)\|_{L^p(\mathbb{R})} \leqslant \eta(p, r)\, w^{-r}\, \|f^{(r)}\|_{L^p(\mathbb{R})}. \tag{3.25}$$

Proof. By Proposition 3.1 and the above remark, the inequality (3.25) is true for the case $w = 1$. For the general case $w > 0$, we make a transform of dilation as follows. Let $g(x) := f(x/w)$. Then one can easily see that $s_{r-1,w}(f; x) = s_{r-1}(g; wx)$. In the following we consider only the case $1 \leqslant p < \infty$. The proof for the case $p = \infty$ is similar. Thus,

$$\|f - s_{r-1,w}(f)\|_{L^p(\mathbb{R})}$$
$$= \left(\int_{\mathbb{R}} |g(wx) - s_{r-1}(g; wx)|^p \, dx \right)^{1/p}$$
$$= \left(\frac{1}{w} \int_{\mathbb{R}} |g(y) - s_{r-1}(g; y)|^p \, dy \right)^{1/p} = w^{-1/p}\, \|g - s_{r-1}(g)\|_{L^p(\mathbb{R})}$$
$$\leqslant w^{-1/p} \eta(p, r)\, \|g^{(r)}\|_{L^p(\mathbb{R})} = w^{-1/p-r} \eta(p, r) \left(\int_{\mathbb{R}} \left| f^{(r)}\left(\frac{y}{w}\right) \right|^p dy \right)^{1/p}$$
$$= w^{-r} \eta(p, r) \left(\int_{\mathbb{R}} |f^{(r)}(x)|^p \, dx \right)^{1/p} = w^{-r} \eta(p, r)\, \|f^{(r)}\|_{L^p(\mathbb{R})}. \quad ∎$$

To get the lower bound, we need the following lemma. Let X be a normed linear space and $A \subset X$. By $b_n(A; X)$ we denote the Bernstein n-width [11] of A in X.

LEMMA 3.5. *Let n and r be positive integers and $p \in [1, \infty]$. Then*

$$b_n(B_p^r(I)_0; L^p(I)) \geqslant b_{n+r}(\tilde{B}_p^r(I); L^p(I)),$$

where $I = [a, b]$ is a finite interval, and $\tilde{B}_p^r(I)$ and $B_p^r(I)_0$ are defined in (3.3) *and* (3.4), *respectively.*

Proof. Given $\varepsilon > 0$, according to the definition of the Bernstein n-width [11], there exist a $\mu > 0$ and a subspace $X_{n+r+1} \subset L^p(I)$ with $\dim X_{n+r+1} = n + r + 1$, such that

$$\mu S(X_{n+r+1}) \subseteq \tilde{B}_p^r(I) \qquad \text{and} \qquad \mu + \varepsilon > b_{b+r}(\tilde{B}_p^r(I); L^p(I)) \geqslant \mu,$$

where $S(X_{n+r+1}) := \{f \in X_{n+r+1} : \|f\|_{L^p(I)} \leqslant 1\}$. Note that from the first containing relation we know that each element of X_{n+r+1} has continuous derivatives up to order $r - 1$. Put

$$X_{n+1}^* := \{f \in X_{n+r+1} : f^{(j)}(a) = 0, j = 0, ..., r-1\}.$$

Then $\dim X_{n+1}^* \geqslant \dim X_{n+r+1} - r = n + 1$ and $\mu S(X_{n+1}^*) \subseteq B_p^r(I)_0$. Therefore

$$b_n(B_p^r(I)_0; L^p(I)) \geqslant \mu > b_{n+r}(\tilde{B}_p^r(I); L^p(I)) - \varepsilon.$$

Letting $\varepsilon \to 0^+$ we conclude the desired inequality. ▮

Proof of Theorem 3.1. We first point out that $\mathscr{S}_{r-1,w} \in \mathscr{T}_w$, where $\mathscr{T}_w$ is the family of spaces defined in Section 2 and $\mathscr{S}_{r-1,w}$ is given in (3.9). In fact, if we consider the B-spline function [2]

$$M_{r,w}(x) := r\left[0, \frac{1}{w}, ..., \frac{r}{w}\right](\cdot - x)_+^{r-1}$$

with $0, 1/w, ..., r/w$ as simple knots, then $M_{r,w}$ has compact support $[0, r/w]$ and $\mathscr{S}_{r-1,w} = \operatorname{span}\{M_{r,w}(\cdot - k/w)\}_{k \in \mathbb{Z}}$. Thus according to Section 2 we see that $\mathscr{S}_{r-1,w} \in \mathscr{T}_w$. Now, observing that $s_{r-1,w}$ (cf. (3.10)) is a linear operator which maps $W_p^r(\mathbb{R}) = \operatorname{span}(B_p^r(\mathbb{R}))$ into $\mathscr{S}_{r-1,w}$, by Proposition 3.2, the definition of ∞-linear width, and the inequality (2.6), we obtain

$$\begin{aligned} d_w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) &\leqslant \delta_w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) \\ &\leqslant \sup_{f \in B_p^r(\mathbb{R})} \|f - s_{r-1,w}(f)\|_{L^p(\mathbb{R})} \leqslant \eta(p, r) w^{-r}. \quad (3.26) \end{aligned}$$

To show that equality holds in (3.26), it remains to prove that $d_w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) \geqslant \eta(p, r) w^{-r}$. By the definition of the ∞-K width, it is sufficient to demonstrate

$$E(B_p^r(\mathbb{R}); S)_p := \sup_{f \in \tilde{B}_p^r(\mathbb{R})} \inf_{g \in S} \|f - g\|_{L^p(\mathbb{R})} \geqslant \eta(p, r) w^{-r}, \qquad \text{for all } \ S \in \mathscr{T}_w. \tag{3.27}$$

Let $\varepsilon > 0$ and $S \in \mathscr{T}_w$. Without loss of generality we can assume that $\dim S = \infty$. From (2.1) we can find a sequence $\{a_k\}_{k=1}^{\infty}$ of positive numbers satisfying $a_k \to \infty$ as $k \to \infty$ such that

$$n_k := \dim S|_{[-a_k, a_k]} \leqslant 2(1+\varepsilon)\, w a_k, \qquad k = 1, 2, \ldots. \tag{3.28}$$

Set $I_k := [-a_k, a_k]$. As we pointed out at the beginning of this section, $B_p^r(I_k)_0$ can be viewed as a subset of $B_p^r(\mathbb{R})$. Thus, by definition we have

$$\begin{aligned} E(B_p^r(\mathbb{R}); S)_p &\geqslant \sup_{f \in \tilde{B}_p^r(I_k)_0} \inf_{g \in S} \|f - g\|_{L^p(\mathbb{R})} \\ &\geqslant \sup_{f \in \tilde{B}_p^r(I_k)_0} \inf_{g \in S|_{I_k}} \|f - g\|_{L^p(I_k)} \geqslant d_{n_k}(B_p^r(I_k)_0; L^p(I_k)), \end{aligned} \tag{3.29}$$

where the last inequality follows from the definition of the Kolmogorov n-width $d_n(A; X)$ and (3.28). By the fact that $d_n(B_p^r(I_k)_0; L^p(I_k)) \geqslant b_n(B_p^r(I_k)_0; L^p(I_k))$ and Lemma 3.5 we get

$$\begin{aligned} E(B_p^r(\mathbb{R}); S)_p &\geqslant b_{n_k}(B_p^r(I_k)_0; L^p(I_k)) \\ &\geqslant b_{n_k+r}(\tilde{B}_p^r(I_k); L^p(I_k)) \\ &= \left(\frac{a_k}{\pi}\right)^r b_{n_k+r}(B_p^r([-\pi, \pi]); L^p([-\pi, \pi])) \\ &\geqslant (\pi w)^{-r}(1+\varepsilon)^{-r} 2^{-r} n_k^r b_{n_k+r}(\tilde{B}_p^r([-\pi, \pi]); L^p([-\pi, \pi])), \end{aligned} \tag{3.30}$$

where the equality follows from a transform of scale of variable argument in the definition of the Bernstein n-width and the last inequality follows from (3.28). For the case $1 < p < \infty$ we know from Chen and Li [5] that

$$\begin{aligned} &\lim_{n \to \infty} 2^{-r} n^r b_{n+r}(\tilde{B}_p^r([-\pi, \pi]); L^p([-\pi, \pi])) \\ &\quad = \lim_{n \to \infty} 2^{-r} \left(\frac{n}{n+r}\right)^r (n+r)^r b_{n+r}(\tilde{B}_p^r([-\pi, \pi]); L^p([-\pi, \pi])) \\ &\quad = \sup\{\|f\|_{L^p[-\pi,\pi]} : f \in \tilde{B}_p^r([-\pi, \pi]) \text{ and } f(-\cdot) = f(\cdot) = -f(\cdot + \pi)\} \\ &\quad = \pi^r \sup\{\|f\|_{L^p[-1,1]} : f \in \tilde{B}_p^r([-1, 1]) \text{ and } f(-\cdot) = f(\cdot) = -f(\cdot + 1)\} \\ &\quad = \pi^r \eta(p, r). \end{aligned} \tag{3.31}$$

From the monograph [11, pp. 133, 180, and 183] we see that the above strong asymptotic relation is also true in the cases $p=1$ and $p=\infty$. Therefore, letting $k\to\infty$ in (3.30) and noticing that $n_k\to\dim S=\infty$, we conclude that

$$E(B_p^r(\mathbb{R}); S)_p \geqslant (1+\varepsilon)^{-r}\eta(p,r)w^{-r}.$$

Since $\varepsilon>0$ is arbitrary, (3.27) follows, and therefore $d_w(B_p^r(\mathbb{R}); L^p(\mathbb{R}))\geqslant \eta(p,r)w^{-r}$. Combining this inequality with (3.26) gives

$$\begin{aligned} d_w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) &= \delta_w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) \\ &= \sup_{f\in B_p^r(\mathbb{R})} \|f-s_{r-1,w}(f)\|_{L^p(\mathbb{R})} = \eta(p,r)w^{-r}. \end{aligned} \tag{3.32}$$

To complete the proof of Theorem 3.1, we must show

$$d^w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) = \sup_{f\in B_p^r(\mathbb{R})\cap \operatorname{Ker} T^*} \|f\|_{L^p(\mathbb{R})} = \eta(p,r)w^{-r}, \tag{3.33}$$

where $\operatorname{Ker} T^*$ is given in (3.11). For $A=B_p^r(\mathbb{R})$ we have $Y_A(\mathbb{R}) := \operatorname{span}(A) = W_p^r(\mathbb{R})$. By $(W_p^r(\mathbb{R}))'$ we denote the dual space of $W_p^r(\mathbb{R})$, and for ease of notation, we set

$$\begin{aligned} \Theta_w &:= \Theta_w(A) \\ &= \left\{ T=\{\tau_j\}_{j\in\mathbb{Z}} : \tau_j\in (W_p^r(\mathbb{R}))',\, j\in\mathbb{Z},\, \liminf_{a\to+\infty} \frac{1}{2a}\operatorname{card}(T|_{[-a,a]}) \leqslant w \right\}. \end{aligned} \tag{3.34}$$

Again, let $\varepsilon>0$ and $T=\{\tau_j\}_{j\in\mathbb{Z}}\in\Theta_w$. By definition we can find a sequence $\{a_k\}_{k=1}^\infty$ of positive numbers with $a_k\to\infty$ as $k\to\infty$, such that

$$n_k := \operatorname{card}(T|_{[-a_k,a_k]}) \leqslant 2a_k w(1+\varepsilon), \qquad k=1,2,\ldots. \tag{3.35}$$

Put $I_k := [-a_k, a_k]$. Then we have

$$\begin{aligned} &\sup_{f\in B_p^r(\mathbb{R})\cap \operatorname{Ker} T} \|f\|_{L^p(\mathbb{R})} \\ &\quad\geqslant \sup_{f\in B_p^r(I_k)_0\cap \operatorname{Ker} T} \|f\|_{L^p(\mathbb{R})} \\ &\quad= \sup_{f\in B_p^r(I_k)_0\cap \operatorname{Ker} T|_{I_k}} \|f\|_{L^p(I_k)} \geqslant d^{n_k}(B_p^r(I_k)_0; W_p^r(I_k)_0), \end{aligned} \tag{3.36}$$

where the last inequality follows from the definition of the Gel'fand n-width $d^n(A; X)$ and the definition of n_k. Note that we can view the continuous linear functionals in $T|_{I_k}$ as elements of $(W_p^r(I_k)_0)'$, where $W_p^r(I_k)_0 :=$

$\text{span}(B_p^r(I_k)_0)$ is a subspace of $L^p(I_k)$ with norm $\|\cdot\|_{L^p(I_k)}$. By well known properties [11] of the Gel'fand n-width, we have

$$d^n(B_p^r(I_k)_0;\, W_p^r(I_k)_0) = d^n(B_p^r(I_k)_0;\, L_p^r(I_k)) \geqslant b_n(B_p^r(I_k)_0;\, L_p^r(I_k)). \tag{3.37}$$

As an analog to the previous deduction (cf. (3.28)–(3.31)), from (3.35), (3.36), and (3.37) we can conclude that

$$\sup_{f \in B_p^r(\mathbb{R}) \cap \operatorname{Ker} T} \|f\|_{L^p(\mathbb{R})} \geqslant (1+\varepsilon)^{-r} \eta(p, r) w^{-r}.$$

Since $\varepsilon > 0$ and $T \in \Theta_w$ are arbitrary, it follows that

$$d^w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) = \inf_{T \in \Theta_w} \sup_{f \in B_p^r(\mathbb{R}) \cap \operatorname{Ker} T} \|f\|_{L^p(\mathbb{R})} \geqslant \eta(p, r) w^{-r}. \tag{3.38}$$

To prove the converse inequality, we consider $T^* = \{\tau_j^*\}_{j \in \mathbb{Z}}$, where $\tau_j^*(f) = f(j/w)$, $j \in \mathbb{Z}$. Then $\operatorname{Ker} T^*$ is given in (3.11). Note that $\operatorname{supp} \tau_j = \{j/w\}$, and, therefore, $T^* \in \Theta_w$. Hence

$$\begin{aligned} &d^w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) \\ &\quad\leqslant \sup_{f \in B_p^r(\mathbb{R}) \cap \operatorname{Ker} T^*} \|f\|_{L^p(\mathbb{R})} \\ &\quad= \sup\left\{ \|f\|_{L^p(\mathbb{R})} : f \in B_p^r(\mathbb{R}),\, f\left(\frac{k}{w}\right) = 0, \text{ all } k \in \mathbb{Z} \right\} \\ &\quad= \sup\left\{ \|f\|_{L^p(\mathbb{R})} : f \in B_p^r(\mathbb{R}),\, f\left(\frac{k+\alpha_r}{w}\right) = 0, \text{ all } k \in \mathbb{Z} \right\} \\ &\quad\leqslant \sup\{ \|f - s_{r-1,w}(f)\|_{L^p(\mathbb{R})} : f \in B_p^r(\mathbb{R}) \} \leqslant \eta(p, r) w^{-r}, \end{aligned}$$

where the last inequality follows from Proposition 3.2. Hence (3.33) follows from (3.38) and (3.39). Finally, by (3.32) and (3.33) we finish our proof for Theorem 3.1. ∎

4. An Application to Optimal Recovery for $B_p^r(\mathbb{R})$ in $L^p(\mathbb{R})$

Let the Sobolev function classes $W_p^r(\mathbb{R})$ and $B_p^r(\mathbb{R})$ be given as in Section 3. In this section we want to study the problem of optimal recovery for $B_p^r(\mathbb{R})$ in $L^p(\mathbb{R})$ with infinite many function values as information. We will provide a solution to this problem by using ∞-widths.

Let us now formulate the problem of optimal recovery in the sense of Micchelli and Rivlin [10]. For $w > 0$, we define

$$\hat{\Theta}_w := \left\{ \xi = \{\xi_j\}_{j \in \mathbb{Z}} : \xi_j < \xi_{j+1},\, j \in \mathbb{Z},\, \liminf_{a \to +\infty} \frac{1}{2a} \operatorname{card}(\xi \cap [-a, a]) \leqslant w \right\}. \tag{4.1}$$

For each $\xi \in \hat{\Theta}_w$, we can determine a mapping $I_\xi\colon W_p^r(\mathbb{R}) \to \mathbb{R}^{\mathbb{Z}}$, $I_\xi(f) := (f(\xi_j))_{j\in\mathbb{Z}}$. We say that I_ξ is an information operator. An arbitrary mapping $A\colon I_\xi(B_p^r(\mathbb{R})) \to L^p(\mathbb{R})$ is called an algorithm. We consider the approximation problem $\sup\{\|f - A(I_\xi(f)\|_{L^p(\mathbb{R})}: f \in B_p^r(\mathbb{R})\}$. Taking the infimum over the expression for all possible algorithms leads to the intrinsic error

$$E(B_p^r(\mathbb{R}); \xi) := \inf_A \sup_{f\in B_p^r(\mathbb{R})} \|f - A(I_\xi(f))\|_{L^p(\mathbb{R})}. \tag{4.2}$$

To find the optimal set of sampling points in $\hat{\Theta}_w$, we also want to study

$$E(B_p^r(\mathbb{R}); \hat{\Theta}_w) := \inf_{\xi\in\hat{\Theta}_w} E(B_p^r(\mathbb{R}); \xi). \tag{4.3}$$

The problems of optimal recovery of this type were initiated by Sun [13] in the case $p = \infty$. Since then several results for cases $p = 1$, $p = 2$, and other function classes have been obtained. The interested readers may refer to [8, 15, 14]. Here we will solve the above problems in the general case $p \in (1, \infty)\backslash\{2\}$.

Since $B_p^r(\mathbb{R})$ is symmetric about the origin (i.e., $f \in B_p^r(\mathbb{R})$ implies $-f \in B_p^r(\mathbb{R})$), it follows from [10] that

$$E(B_p^r(\mathbb{R}); \xi) \geqslant \sup\{\|f\|_{L^p(\mathbb{R})}: f \in B_p^r(\mathbb{R}), f(\xi_j) = 0, j \in \mathbb{Z}\}. \tag{4.4}$$

For $\xi \in \hat{\Theta}_w$, let $\tau_j \in (W_p^r(\mathbb{R}))'$ be defined by $\tau_j(f) = f(\xi_j)$, $j \in \mathbb{Z}$. Then one can easily verify that $T := \{\tau_j\}_{j\in\mathbb{Z}} \in \Theta_w$, where Θ_w is defined by (3.34). Thus, according to the definition of the ∞-G width and Theorem 3.1 we have

$$\begin{aligned}\sup\{\|f\|_{L^p(\mathbb{R})}&: f \in B_p^r(\mathbb{R}), f(\xi_j) = 0, j \in \mathbb{Z}\}\\ &\geqslant d^w(B_p^r(\mathbb{R}); L^p(\mathbb{R})) = \eta(p, r)w^{-r}.\end{aligned} \tag{4.5}$$

From (4.3), (4.4), and (4.5) we obtain

$$\begin{aligned}E(B_p^r(\mathbb{R}); \hat{\Theta}_w) &\geqslant \inf_{\xi\in\hat{\Theta}_w} \sup\{\|f\|_{L^p(\mathbb{R})}: f \in B_p^r(\mathbb{R}), f(\xi_j) = 0, j \in \mathbb{Z}\}\\ &\geqslant \eta(p, r)w^{-r}.\end{aligned} \tag{4.6}$$

On the other hand, for $\xi^* := \{(k + \alpha_r)/w\}_{k\in\mathbb{Z}} \in \hat{\Theta}_w$, by Proposition 3.2 we have

$$\begin{aligned}E(B_p^r(\mathbb{R}); \hat{\Theta}_w) &\leqslant E(B_p^r(\mathbb{R}); \xi^*)\\ &\leqslant \sup_{f\in B_p^r(\mathbb{R})} \|f - s_{r-1,w}(f)\|_{L^p(\mathbb{R})} \leqslant \eta(p, r)w^{-r}.\end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) gives the following:

THEOREM 4.1. *Let r be a positive integer, $w>0$, $p\in(1,\infty)$, and the interpolation operator $s_{r-1,w}$ be defined by* (3.10). *Then*

$$E(B_p^r(\mathbb{R});\hat{\Theta}_w)=E(B_p^r(\mathbb{R});\xi^*)$$
$$=\sup_{f\in B_p^r(\mathbb{R})}\|f-s_{r-1,w}(f)\|_{L^p(\mathbb{R})}=\eta(p,r)w^{-r}.$$

That is, $\xi^*=\{(k+\alpha_r)/w\}_{k\in\mathbb{Z}}$ is an optimal set of sampling points and $s_{r-1,w}$ is an optimal algorithm which realizes $E(B_p^r(\mathbb{R});\hat{\Theta}_w)$.

Remark. The above results are also valid in the cases $p=1$ and $p=\infty$.

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